Local-dephasing-induced entanglement sudden death in two-component finite-dimensional systems

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Abstract

Entanglement sudden death (ESD), the complete loss of entanglement in finite time, is demonstrated to occur in a class of bipartite states of qu-d-it pairs of any finite dimension $d > 2$, when prepared in so-called 'isotropic states' and subject to multi-local dephasing noise alone. This extends previous results for qubit pairs [T. Yu, J. H. Eberly, Phys. Rev. Lett. **97**, 140403 (2006)] to all qu-d-it pairs with $d > 2$.

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Entanglement is perhaps the most quantum mechanical property a physical system can possess. The behavior of entanglement under the influence of environmental noise is important to quantum measurements and enables powerful quantum computations [1, 2]. Noise, even acting locally or on phases alone, may cause not only state decoherence but also state disentanglement [3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Indeed, recent work has shown that even weak local noise acting on bipartite states of infinite-dimensional systems, pairs of qubits, and qubit–qutrit systems can lead to *entanglement sudden death* (ESD), a total loss of state entanglement in finite time with generic decoherence taking place only asymptotically [8, 9, 11, 12]. Here, we extend these results, showing the existence of weak local dephasing noise induced ESD in bipartite isotropic qudit-qudit states [14] for every finite dimension $d > 2$ using the entanglement of formation E_{f} as a measure of entanglement.

The isotropic states are those invariant under transformations of the form $U \otimes U^*$, where U is unitary [14]. The general $d \times d$-dimensional isotropic states $\rho_{\mathrm{iso}}(d)$ are convex combinations of a maximally mixed state $(d^{-2})\mathbb{I}_{d^2}$ and a maximally entangled projector $P(|\Psi(d)\rangle) \equiv |\Psi(d)\rangle\langle\Psi(d)|$:

$$\rho_{\mathrm{iso}}(d) = \left(\frac{1-F}{d^2-1}\right)\mathbb{I}_{d^2} + \left(\frac{Fd^2-1}{d^2-1}\right)P(|\Psi(d)\rangle), \tag{1}$$

where $d > 1$, $\mathbb{I}_{d^2}$ is the $d^2 \times d^2$ identity matrix, $|\Psi(d)\rangle = (1/\sqrt{d})\sum_{i=1}^{d}|i\rangle|i\rangle$; the fidelity $F(\rho_{\mathrm{iso}}(d), P(|\Psi(d)\rangle)) = \mathrm{tr}\,(\rho_{\mathrm{iso}}(d)P(|\Psi(d)\rangle))$ [13], which is bounded by 0 and 1 and appears self-consistently in the formal definition of isotropic states [14], proves convenient for our study of disentanglement. The state $\rho_{\mathrm{iso}}(d)$ is separable if and only if $F\,(\rho_{\mathrm{iso}}(d), P(|\Psi\rangle)) \leq F_{\mathrm{critical}}(d) \equiv d^{-1}$, according to the standard measure of entanglement, the entanglement of formation: for the isotropic states $\rho_{\mathrm{iso}}(d)$ for $d > 2$,

$$E_{\mathrm{f}}\left(\rho_{\mathrm{iso}}\right) = \begin{cases} 0, & F \leq \frac{1}{d}\ , \\ R_{1,d-1}\left(F\right), & F \in \left[\frac{1}{d}, \frac{4(d-1)}{d^2}\right]\ , \\ \frac{d\log(d-1)}{d-2}\left(F-1\right) + \log d, & F \in \left[\frac{4(d-1)}{d^2}, 1\right]\ , \end{cases} \tag{2}$$

where $R_{1,d-1}\left(F\right) = H_2\left(\xi(F)\right) + \left[1-\xi(F)\right]\log_2(d-1)$, $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$, and $\xi(F) = \frac{1}{d}\left[\sqrt{F} + \sqrt{(d-1)(1-F)}\right]^2$ [15, 16]. We have chosen to use the entanglement of formation from among the various entanglement measures [15, 16, 17, 18, 19, 20, 21, 22, 23]. Less standard measures, such as concurrence and negativity, have typically been used to study ESD. The concurrence is a readily calculated mixed-state entanglement measure for

2×2 systems [21]. The negativity can be used for mixed states of 2×2 and 2×3 systems [7, 22]. For larger finite-dimensional bipartite systems, there is no known general closed form expression for entanglement applicable to all states. However, we can use the above specific form for the entanglement of formation that is valid for arbitrary isotropic mixed states of such systems, our case. Eq. 2 is valid for $d > 2$ (although it does not apply in the case $d = 2$); Terhal and Vollbrecht showed its validity for $d = 3$ and conjectured it for arbitrary $d \geq 3$ [16], a conjecture later proven to be true by Fei and Li-Jost [15].

For ESD to occur, entanglement must be initially positive and go to zero in finite time. To demonstrate that ESD from an isotropic initial state $\rho_{\rm iso}(d)$, it suffices to show that the fidelity $F\left(\rho_{\rm iso}(d), P(|\Psi(d)\rangle\right)$ is initially above $F_{\rm critical} = d^{-1}$ and later drops to that value at some $t < \infty$. Our interest is in states of qudit pairs with $d > 2$. We begin with a simple model illustrating basic dephasing, based on which conclusions about the general case of isotropic noise, wherein initially isotropic states are certain to remain isotropic, are later drawn. The general time-evolved open-system density matrix expressible in the operator-sum decomposition of an open-system evolution is the completely positive trace preserving (CPTP) map $\rho\left(t\right) = \mathcal{E}\left(\rho\left(0\right)\right) = \sum_{\mu} K_{\mu}^{\dagger}\left(t\right)\rho\left(0\right)K_{\mu}\left(t\right)$; the operators $\{K_{\mu}(t)\}$ satisfy the completeness condition $\sum_{\mu} K_{\mu}^{\dagger}(t)K_{\mu}(t) = \mathbb{I}$ and the trace preserving condition $\sum_{\mu} K_{\mu}(t)K_{\mu}^{\dagger}(t) = \mathbb{I}$, and represent the influence of statistical noise [1, 3, 4]. For our model of multi-local dephasing noise acting on a bipartite state, $K_{\mu}(t) = D_j(t)E_i(t)$:

$$\rho\left(t\right) = \mathcal{E}\left(\rho\left(0\right)\right) = \sum_{i,j=1}^{2} D_j^{\dagger}\left(t\right)E_i^{\dagger}\left(t\right)\rho\left(0\right)E_i\left(t\right)D_j\left(t\right), \tag{3}$$

where $E_i(t)$ and $D_j(t)$ correspond to local dephasing noise components acting on the first and second qudit, respectively, and individually satisfy the above conditions. We take these to be of the specific forms

$$E_1(t) \; = \; \mathrm{diag}(1, \gamma_{\rm A}, \gamma_{\rm A}, \ldots, \gamma_{\rm A}) \otimes \mathbb{I}_d \; , E_2(t) = \mathrm{diag}(0, \omega_{\rm A}, \omega_{\rm A}, \ldots, \omega_{\rm A}) \otimes \mathbb{I}_d \; , \tag{4}$$

$$D_1(t) \; = \; \mathbb{I}_d \otimes \mathrm{diag}(1, \gamma_{\rm B}, \gamma_{\rm B}, \ldots, \gamma_{\rm B}) \; , D_2(t) = \mathbb{I}_d \otimes \mathrm{diag}(0, \omega_{\rm B}, \omega_{\rm B}, \ldots, \omega_{\rm B}) \; , \tag{5}$$

where $\gamma_{\rm A}\left(t\right) = e^{-\Gamma_{\rm A}t/2}$, $\gamma_{\rm B}\left(t\right) = e^{-\Gamma_{\rm B}t/2}$, $\omega_{\rm A}\left(t\right) = \sqrt{1-\gamma_{\rm A}^2(t)}$, and $\omega_{\rm B}\left(t\right) = \sqrt{1-\gamma_{\rm B}^2(t)}$. For simplicity, these noise parameters are chosen so that the rate of dephasing from state k relative to the state 1 are equal, that is, $\Gamma_{\rm A} = \Gamma_{\rm B} = \Gamma$, and hence $\gamma_{\rm A}(t) = \gamma_{\rm B}(t) = \gamma(t)$, although subscripts may occasionally appear for clarity and the time-dependence of γ's may

be implicit. This simple model generalizes well to the case where dephasing occurs between all states of our basis.

The initial value F_0 of the time-dependent fidelity $F\left(\rho(d,t), P(|\Psi(d,t)\rangle)\right) = \operatorname{tr}\left(\rho(d,t)P(|\Psi(d,t)\rangle)\right)$ of the time-evolved states, for each value of d, has an corresponding to the choice of initial isotropic state, $\rho(d,0)$. The initial state is

$$\rho(d,0) = \epsilon \mathbb{I}_{d^2} + \zeta P(|\Psi(d)\rangle) \;, \tag{6}$$

where $\epsilon \equiv \frac{1-F_0}{d^2-1}$ and $\zeta \equiv \frac{F_0 d^2-1}{d^2-1}$. The first term contributes a summand of ϵ to each element of the density matrix diagonal and nothing elsewhere, since it is a multiple of $\mathbb{I}_{d^2}$; the second term, which involves $P(|\Psi(d)\rangle)$, contributes ζd^{-1} at positions $(\mathrm{row},\mathrm{col}) = ((j-1)d+j,(k-1)d+k)$ for $1 \leq j,k \leq d$ and zeros elsewhere. Here, the joint-system density matrix is studied in the tensor product of the individual subsystem bases $|1\rangle = (10\ldots0)^{\mathrm{T}}, |2\rangle = (010\ldots0)^{\mathrm{T}}, \ldots, |d\rangle = (0\ldots01)^{\mathrm{T}}$. The initial state density matrix in explicit matrix form is

$$\begin{aligned}
\rho(d,0) &= \epsilon \mathbb{I}_{d^2} + \zeta P(|\Psi(d)\rangle) && (7)\\
&= \operatorname{diag}(\epsilon,\epsilon,\ldots,\epsilon) + \frac{1}{d}\begin{pmatrix} \mathcal{M}_I & \cdots & \mathcal{M}_I & \mathcal{M}_{II} \\ \mathcal{M}_I & \cdots & \mathcal{M}_I & \mathcal{M}_{II} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{M}_I & \cdots & \mathcal{M}_I & \mathcal{M}_{II} \\ \mathcal{M}_{III} & \cdots & \mathcal{M}_{III} & \mathcal{M}_{IV} \end{pmatrix}, && (8)
\end{aligned}$$

$$\mathcal{M}_I = \begin{pmatrix} \zeta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; \mathcal{M}_{II} = \begin{pmatrix} \zeta & 0 & \cdots & 0 & \zeta \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\mathcal{M}_{III} = \begin{pmatrix} \zeta & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \zeta & 0 & \cdots & 0 & 0 \end{pmatrix}, \; \mathcal{M}_{IV} = \begin{pmatrix} \zeta & 0 & \cdots & 0 & \zeta \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \zeta & 0 & \cdots & 0 & \zeta \end{pmatrix},$$

wherein there are $d-2$ of the $\mathcal{M}_I$ $(d+1)\times(d+1)$ Hermitian matrices on the rows and columns, $d-2$ of the $\mathcal{M}_{II}$ $(d+1)\times(d+2)$ on the last column, $d-2$ of the $\mathcal{M}_{III}$ $(d+2)\times(d+1)$ on the last row, $\mathcal{M}_{IV}$ is a $(d+2)\times(d+2)$ Hermitian matrix; $\mathcal{M}_{II}=\mathcal{M}_{III}^{\dagger}$.

The time-evolved density matrix $\rho\,(d,t)=\mathcal{E}\,(\rho\,(d,0))$, that is, the solution of Eq. 3 for $t\geq 0$, consists of decaying factors $\tilde{\gamma}(t)$ multiplying the elements of $\rho(d,0)$ at $(\text{row},1)=((j-1)d+j,1)$ for $2\leq j\leq d$ and at $(1,\text{col})=(1,(k-1)d+k)$ for $2\leq k\leq d$, where $\tilde{\gamma}(t)$ represents $\gamma_{\text{A}}(t)$, $\gamma_{\text{B}}(t)$, and $\gamma_{\text{A}}(t)\gamma_{\text{B}}(t)$ in the cases of local noise acting on A alone, B alone, and on both, respectively; that is, decaying terms appear in the first row and first column only, because in the simple noise model we consider for now there is dephasing of the k^{th} state for $2\leq k\leq d$ relative to the ground state $k=1$, but no dephasing between other basis states. Because we are not concerned with precisely when full disentanglement occurs, only that it *does* occur in finite time, specific decay rates appearing in the $\tilde{\gamma}(t)$ and from hereon collectively designated $\tilde{\Gamma}$, are not crucial—they must only be nonzero. The time-dependent state is

$$\begin{aligned}
\rho(d,t) &= \epsilon\mathbb{I}_{d^2}+\zeta P(|\Psi(d,t)\rangle) \qquad (9)\\
&= \text{diag}(\epsilon,\epsilon,\ldots,\epsilon)+\frac{1}{d}\begin{pmatrix} \mathcal{M}_I & \mathcal{M}_I\tilde{\gamma} & \cdots & \mathcal{M}_I\tilde{\gamma} & \mathcal{M}_{II}\tilde{\gamma}\\ \mathcal{M}_I\tilde{\gamma} & \mathcal{M}_I & \cdots & \mathcal{M}_I & \mathcal{M}_{II}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathcal{M}_I\tilde{\gamma} & \mathcal{M}_I & \cdots & \mathcal{M}_I & \mathcal{M}_{II}\\ \mathcal{M}_{III}\tilde{\gamma} & \mathcal{M}_{III} & \cdots & \mathcal{M}_{III} & \mathcal{M}_{IV}\end{pmatrix}. \qquad (10)
\end{aligned}$$

The bipartite system state will remain partially coherent for all finite times because all off-diagonal elements persist for all finite times; only in the limit $t\to\infty$ is there full decoherence between the ground state and every other state. However, as we now show, there still is complete loss of entanglement in *finite* time for a range of initial isotropic states. It is valuable to note here that the production of such states and their non-local measurement may be experimentally challenging.

To see that complete disentanglement does indeed take place in finite time, we first find the time-dependent fidelity $F\,(\rho(d,t),P(|\Psi\rangle))\;=\;\text{tr}\,(\rho(d,t),P(|\Psi\rangle))$. The argument $\rho(d,t)P(|\Psi\rangle)=\boldsymbol{M}$ has three distinct sorts of terms, C_1, C_2, and C_3, having specific forms which we describe in turn and then evaluate. The sole C_1 term appears at $\boldsymbol{M}_{1,1}$; C_2 terms appear at $\boldsymbol{M}_{\text{row},\text{col}}$ with $(\text{row},\text{col})=((j-1)d+j,(k-1)d+k)$ for $2\leq j,k\leq d$, $\delta_{jk}=1$; C_3

consists of the remaining terms of the matrix. We designate the values of the terms of sorts C_1, C_2, and C_3, by c_1, c_2, and c_3, respectively. The fidelity has nontrivial contributions only from terms from the first and second of these classes, of which there are numbers N_1 and N_2, respectively. In the above simple model, C_1 consists of the single term appearing as $\boldsymbol{M}_{1,1}$, being the inner product of the first row of $\rho(d,t)$ and the first column of $P\left(|\Psi(d,t)\rangle\right)$, taking the value $c_1 = \left(\epsilon + \frac{\zeta}{d}\right)\left(\frac{1}{d}\right) + \left(\frac{\zeta}{d}\tilde{\gamma}(t)\right)\left(\frac{1}{d}\right)(d-1)$, and $N_1 = 1$. C_2 terms are those appearing at $\boldsymbol{M}_{\mathrm{row,col}}$ for $(\mathrm{row},\mathrm{col}) = ((j-1)d+j,(k-1)d+k)$ for $2 \le j,k \le d$ with $\delta_{jk} = 1$, and are inner products, each taking the value $c_2 = \left(\frac{\zeta}{d}\tilde{\gamma}(t)\right)\left(\frac{1}{d}\right) + \left(\epsilon + \frac{\zeta}{d}\right)\left(\frac{1}{d}\right) + \left(\frac{\zeta}{d}\right)\left(\frac{1}{d}\right)(d-2)$, and $N_2 = d-1$. The time-dependent fidelity for this model is thus

$$F\left(\rho(d,t), P(|\Psi(d,t)\rangle)\right) = c_1N_1 + c_2N_2 + c_3N_3 = 2\frac{(d^2F_0-1)\tilde{\gamma}(t) + d^2(d-1)\frac{F_0}{2} + 1}{d^3+d^2} \ , \quad (11)$$

which is determined by the initial state fidelity F_0, d of the individual qudits, and t.

Recall that $F(d,t)$ calculated above must initially be above the value $F_{\mathrm{critical}}(d) = d^{-1}$ at and below which isotropic states are separable, that is, the entanglement of formation is zero, and *in finite time* reach that value in order for entanglement sudden death to occur. Note that separability occurs whenever the entanglement is zero independently of the particular entanglement measured used, because this is a defining property any valid entanglement measure. Considering now $G(d,t) \equiv F(d,t) - F_{\mathrm{critical}}(d)$, we show that both $F_0(d) > F_{\mathrm{critical}}(d)$ and this function $G(d,t) = 0$ for some $t < \infty$, for a specific form of $F_0(d)$. Taking the initial fidelities to be $F_0(d) = (d-1)^{-1}$, we have in this simple model

$$G(d,t)\big|_{F_0(d)} = F(d,t)\big|_{F_0(d)=(d-1)^{-1}} - F_{\mathrm{critical}}(d) = \frac{2(d^2-d+1)\tilde{\gamma}(t) - (d-1)(d-2)}{d^2(d^2-1)} \ , \quad (12)$$

which is $[d(d-1)]^{-1} > 0$ at $t = 0$ and is zero at time $t = (2/\tilde{\Gamma})\ln[2(d^2-d+1)/(d-1)(d-2)]$. Recall that in the noise model considered thus far, dephasing noise occurs only between the ground state $k = 1$ and the k^{th} basis state (for $k = 2,3,\ldots,d$). This model is neither the simplest case of local dephasing, wherein there is dephasing between only two particular local basis states, nor is it the most general case wherein dephasing occurs between all pairs of states within each subsystem. Under it, initially isotropic states become anisotropic. However, the expressions resulting from this noise simply generalize to the case of the noise model inducing dephasing between *all* local basis states, in which isotropic states remain isotropic, that is descriptive of what would be encountered in a highly random local phase-noise environment: the solution for the time-dependent density matrix differs from the

above solution only by a $\tilde{\gamma}(t)$ decay factor in *each* nonzero off-diagonal element. Because the dephasing noise is isotropic in this general case, the time-evolved states remain isotropic and the resulting fidelity $F(\rho(d,t), P(|\Psi(d,t)\rangle))$ properly determines the entanglement.

The terms of the C_1 and C_3 types contributing to the fidelity are unchanged under this generalization, but the C_2-type terms change: $c_2 = \left(\frac{\zeta}{d}\tilde{\gamma}(t)\right)\left(\frac{1}{d}\right) + \left(\epsilon + \frac{\zeta}{d}\right)\left(\frac{1}{d}\right) + \left(\frac{\zeta}{d}\right)\left(\frac{1}{d}\right)(d-2) \rightarrow \left(\frac{\zeta}{d}\tilde{\gamma}(t)\right)\left(\frac{1}{d}\right) + \left(\epsilon + \frac{\zeta}{d}\right)\left(\frac{1}{d}\right) + \left(\frac{\zeta}{d}\tilde{\gamma}(t)\right)\left(\frac{1}{d}\right)(d-2)$. An "additional" factor of $\tilde{\gamma}(t)$ appears in the third contribution. The effect on the functions $F(d,t)$ and $G(d,t)$ of this extra decay factor is only a more rapid decrease because the third term also decays to zero. There is no qualitative effect on the behavior of F and G: the fidelity only decreases more rapidly. However, the resulting fidelity now determines the entanglement E_{f}. Hence, ESD occurs for qudit-qudit systems for all finite qudit-space dimensions d greater than 2, when initially prepared in appropriate entangled isotropic states subject to dephasing noise alone. It continues to be exhibited for values of d large like the infinite-dimensional bipartite systems studied in [8].

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